MA520 Introduction to Financial Mathematics

Homework 2 (with Answers)

Due: Monday, December12, 2011 at 11:59pm on ecampus

Name_________________________

Student ID______________________

Answer all question and
Answer each question on the page they are asked.

Show all work

Martingales, Markov, Stochastic, Ito processes

If the Philadelphia Phillies currently are at .600 (they have won 60% of their games so far) give a statement about the remainder of the Phillies games that would make the season (with respect to winning percentage) a

a) Martingale process (hint: the Phillies will win 60% of their remaining game is not correct)

The probability of the Phillies winning the next game is always equal to their current winning percentage.

b) Markov process that is not a Martingale process

Any answer that is not the same as part a and doesn't not require any information other the current percentage.

The Phillies will win 50% of their remaining games. (Markov and not martingale)

The chances of winning the next game is an average of their 3 most recent games) (not Markov, not martingale)

c) Model the win percentage of the reminder of the Phillies season (T-t) as an Ito process (make a sensible choice for $\mu \text{ and } \sigma$ based on the current information you know).

$$dP = 0dt + \sigma(t)dw$$

Notice that there is no non-stochastic term $\mu P dt$ because the present winning percentage P is what we are modeling unlike when we are middling a stock price which is expected to go up, a baseball teams win percentage is a martingale process and therefore is expected to stay the same.

Notice that there is no P in the stochastic term. For modeling stock prices there variance would be multiplied against the current price of the stock that is not much it would vary in dollars (not percentages) away from the current price.

In baseball, the higher your win percentage is will not multiply the effect of your variance. That is higher percentage winning team do not have a higher variance for their upcoming games. Their variance stays the same a low percentage teams.

Now we need to pick a good value for σ. Since the value of the wiener process can be -1 quite often, the outcome of a game is only 1/T in weight for the whole season (assuming a T=162 game season). So a good value for σ would be a function of t. Early in the season a win/loss would greatly vary the win percentage but as time goes on the variance will be less.

$$dP = \left(\frac{T-t}{T}\right) P dw$$

Brownian Motion (Weiner processes)

1) You are starting up a hedge fund and need to raise money from investor before you begin investing money in equities. You fund will follow a wiener process with a drift of +0.5 per quarter and a variance rate of 4.0 per quarter. How much money should you raise (before starting the fund) so that the probability of the fund hitting a zero balance after 1 year is less than 5%.

Suppose the hedge funds initial cash position is x. The probability distribution of the cash position at the end of one year is

$$\Phi\left(x + 4 * 0.5, \sqrt{4} * \sqrt{4}\right) = \Phi(x + 0.2, 4)$$

Where $\Phi(\mu, \sigma)$is a normal probability distribution with mean μ and standard deviation of σ.

The probability of a negative cash position at the end of one year is

$$N\left(-\frac{x + 0.2}{4}\right)$$

Where N(x) is the cumulative probability that a standardized normal variable (with mean zero and standard deviation 1 is less than x.

From the normal distribution tables we have

$$N\left(-\frac{x + 0.2}{4}\right) = 0.05$$

When

$$-\frac{x + 2.0}{4} = -1.6449$$

x = 4.5796 Millions

So you would need to start with $4.58 Million dollars to have a less that 5% chance of zeroing out the fund after 1 year.

Ito's Lemma

1) Suppose that x is the YTM (continuous compounding) on a zero-coupon bond that pays off $1 at time T. Assume that x follows the following Ito process

$$dx = a(x_0 - x)dt + sxdw$$

Where a, x_0, and s are positive constants and dw is a Wiener process. What is the process for the Bond price?

The process followed by B, the bond price, is from Ito's lemma:

$$dB = \left[\frac{\partial B}{\partial x} a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2} s^2 x^2\right] dt + \frac{\partial B}{\partial x} sxdw$$

Since: $B = e^{-x(T-t)}$

The required partial derivatives are

$$\frac{\partial B}{\partial x} = -(T-t)e^{-x(T-t)} = -(T-t)B$$

$$\frac{\partial B}{\partial t} = xe^{-x(T-t)} = xB$$

$$\frac{\partial^2 B}{\partial x^2} = (T-t)^2 e^{-x(T-t)} = (T-t)^2 B$$

Hence:

$$db = \left[-a(x_0 - x)(T-t) + x + \frac{1}{2}s^2x^2(T-t)^2\right] Bdt + sx(T-t)Bdw$$

Black-Scholes

1) What is the price of a European put option on a non-dividend-paying stock when the stock price is $69, the strike price is $70, the risk-free interest rate is 5% per year, the volatility is 35% per year, and the time to maturity is 6 months.

S_0=69, K = 70, r = 0.05, σ = 0.35, T = 0.5

$$d_1 = \frac{\ln\left(\frac{69}{70}\right) + \left(0.05 - \frac{0.35^2}{2}\right)(0.5)}{0.35\sqrt{0.5}} == 0.1666$$

$$d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809$$

The price of a European put is

$$= 70e^{-0.05*0.5}\mathrm{N}(0.0809) - 69\mathrm{N}(-0.1666)$$

$$= 70e^{-0.025} * 0.5323 - 69 * 0.4338 = 6.40$$

2) Find a real stock that doesn't pay a dividend, volatility, interest rate, and calculate it call price for an option in 6 months in the future. See if it matches.

- Pick a non-dividend paying stock like google (GOOG on nasdaq)
- Go to a financial data website like http://www.finance.yahoo.com
- Get the current stock price S
- Pick an call option for about 6 months from now for a particular strike price K
- Estimate the current risk interest rate in the US by going to http://www.treasury.gov and averaging treasury rates for the period you are interested in.
- Calculate the time T as trading days until option expiration
- Estimate the volatility (there is no right way to do this)
    - From www.finance.yahoo.com click on historical data
    - Download to a spreadsheet
    - Create a column called "daily returns" and set it to =LN(price/yesterday price)
    - Create a column called std dev and set it to =STDEV(a good size range of returns)
    - Create a column called "annualized" and set it to =sqrt(252)*sdt dev
    - Create a cell set to =median of a range of annualized